Upper chromatic number of hypergraphs: approximability results

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We study a hypergraph coloring invariant which was first introduced by Berge in the early 1970’s and later independently in different contexts by further authors. A hypergraph \( H = (X, \mathcal{E}) \) is a set system \( \mathcal{E} \) on the underlying vertex set \( X \). An assignment \( \varphi : X \to \mathbb{N} \) is a \( C \)-coloring of \( H \) if each edge \( E \in \mathcal{E} \) has two vertices assigned to the same number (i.e. color). Equivalently, a \( C \)-coloring is a partition of the underlying set \( X \) where no edge \( E \in \mathcal{E} \) is completely sliced by the partition. The upper chromatic number \( \chi(H) \) of \( H \) is the possible maximum number of partition classes which can be achieved under this condition. We use the notation \( n = |X| \) and \( m = |\mathcal{E}| \) for the number of vertices and edges, respectively, in a generic input hypergraph \( H = (X, \mathcal{E}) \).

• For the general case we prove a guaranteed approximation ratio for the difference \( n - \chi(H) \).

A hypertree is a hypergraph \( H = (X, \mathcal{E}) \) for which a ‘host tree’ graph \( T = (X, F) \) exists with the property that each edge of \( H \) induces a connected subgraph in \( T \). We prove the following results on hypertrees:

• \( \chi(H) \) does not have an \( O(n^{1-\epsilon}) \)-approximation in polynomial time (unless \( P = NP \)).
• \( \chi(H) \) cannot be approximated within additive error \( \omega(n) \) in polynomial time, even if each edge of \( H \) contains at most 7 vertices (unless \( P = NP \)).

Our positive result is an algorithm proving the following claim:

• The problems of determining \( \chi(H) \) and finding a \( \chi(H) \)-coloring are fixed-parameter tractable in terms of maximum degree on the class of hypertrees.