Some degree sum and forbidden subgraph conditions for k-contractible edges

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In this note, we deal with finite undirected graphs with neither self-loops nor multiple edges. Let $V_k(G)$ denote the set of vertices of degree k. Let K_n, P_n and C_n denote the complete graph, the path and the cycle on n vertices, respectively. Let K_n^- stand for the graph obtained from K_n by deleting one edge. For graphs G and H, let $G \cup H$ and $G + H$ denote the union of G and H and the join of G and H, respectively. Let k be an integer such that $k \geq 2$ and let G be a connected graph with the connectivity $\kappa(G) = k$ and $|V(G)| \geq k+2$. An edge e of G is said to be k-*contractible* if the contraction of the edge results in a k-connected graph.

Egawa [2] proved the following.

Theorem 1 *Let $k \geq 2$ be an integer, and let G be a k-connected graph with $\delta(G) \geq \lfloor \frac{5}{4}k \rfloor$. Then G has a k-contractible edge, unless $k = 2$ or 3 and G is isomorphic to K_{k+1}.*

If we restrict ourselves to a class of graphs that satisfy some forbidden-subgraph conditions, then we may relax the minimum bound in Theorem 1. In this direction Ando et al. [1] proved the following.

Theorem 2 *For $k \geq 5$, let G be a k-connected graph which contains neither K_5^- nor $5K_1 + P_3$. If $\delta(G) \geq k+1$, then G has a k-contractible edge.*

If $\sum_{x \in V(W)} deg_G(x) \geq mk + 1$ hold for any connected subgraph $W \subseteq G$ with $|W| = m$, then we say that a k-connected graph G satisfies "m-degree-sum-condition". By the definition, G satisfies 1-degree-sum-condition if and only if $\delta(G) \geq k+1$ and G satisfies 2-degree-sum-condition if and only if $V_k(G)$ is independent.

Our new results are the following.

Theorem 3 *Let k be an integer such that $k \geq 5$ and let G be a k-connected graph which has neither $K_2 + (K_1 \cup K_2)$ nor $5K_1 + P_3$. If G satisfies 2-degree-sum condition, then G has a k-contractible edge.*

Theorem 4 *For $k \geq 5$, let G be a k-connected graph which has neither $K_2 + (K_1 \cup K_2)$ nor $K_1 + C_4$. If G satisfies 3-degree-sum condition, then G has a k-contractible edge.*

References

[1] K. Ando, K. Kawarabayashi, Some forbidden subgraph conditions for a graph to have a k-contractible edge, Discrete Math. 267 (2003), 3–11.

[2] Y. Egawa, Contractible edges in n-connected graphs with minimum degree greater than or equal to $[\frac{5n}{4}]$, Graphs Combin. 7 (1991), 15–21.