Polynomial graph invariants from homomorphism numbers

Andrew Goodall
(joint work with Delia Garijo and Jaroslav Nešetřil)

The number of homomorphisms $\hom(G, K_k)$ from a graph G to the complete graph K_k is the value of the chromatic polynomial of G at a positive integer k. This motivates the following (cf. [3]):

Definition 1 *A sequence of graphs* $(H_{\mathbf{k}})$*,* $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$*, is* strongly polynomial *if for every graph* G *there is a polynomial* $p(G; x_1, \ldots, x_h)$ *such that* $\hom(G, H_{\mathbf{k}}) = p(G; k_1, \ldots, k_h)$ *for every* $\mathbf{k} \in \mathbb{N}^h$*.*

Many important graph polynomials $p(G)$ are determined by strongly polynomial sequences of graphs $(H_{\mathbf{k}})$: e.g. [2] the Tutte polynomial, Averbouch–Godlin–Makowsky polynomial [1] (includes the matching polynomial) and Tittmann–Averbouch–Godlin polynomial [4] (includes the independence polynomial).

We give a new construction of strongly polynomial sequences based on coloured rooted tree encodings of graphs (such as cotrees for cographs), which among other things offers a natural generalization of the above polynomials.

In this talk we illustrate this method with the following. We start with a simple graph H given as a spanning subgraph of the closure of a rooted tree T. For each $\mathbf{k} = (k_s : s \in V(T)) \in \mathbb{N}^{|V(T)|}$ we use the tree T to recursively construct a graph $T^{\mathbf{k}}(H)$, in which, for each $s \in V(T)$, we create k_s isomorphic copies of the subtree T_s of T rooted at s, all pendant from the same vertex as T_s, while propagating adjacencies of H in the closure of T to these copies of T_s.

Theorem 1 *The sequence* $(T^{\mathbf{k}}(H))$ *is strongly polynomial.*

Define $\beta(H)$ to be the minimum value of $|V(T)|$ such that H is a subgraph of the closure of $T^{\mathbf{k}}(T)$. For example, $\beta(K_{1,\ell}) = 2$, $\beta(P_{2\ell}) = 2\ell$, $\beta(P_{2\ell-1}) = \ell$, and $\beta(K_\ell) = \ell$. We have tree-depth $\mathrm{td}(H) \leq \beta(H)$ and $\beta(H) = |V(H)|$ if H has no involutive automorphisms.

Theorem 2 *Let* $\mathcal{H}$ *be a family of simple graphs such that* $\{\beta(H) : H \in \mathcal{H}\}$ *is bounded. Then* $\mathcal{H}$ *can be partitioned into a finite number of subsequences of strongly polynomial sequences of graphs.*

References

[1] I. Averbouch, B. Godlin, J.A. Makowsky, A most general edge elimination polynomial, in: H. Broersma et al. (eds.), Graph-theoretic concepts in computer science, WG 2008, LNCS 5344 (2008), 31–42.

[2] D. Garijo, A.J. Goodall, J. Nešetřil, Distinguishing graphs by their left and right homomorphism profiles, European J. Combin. 32 (2011), 1025–1053.

[3] P. de la Harpe, F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, Linear Algebra Appl. 226–228 (1995), 687–722.

[4] P. Tittmann, I. Averbouch, J.A. Makowsky, The enumeration of vertex-induced subgraphs with respect to the number of components, European J. Combin. 32 (2011), 954–974.