Upper chromatic number of hypergraphs: approximability results

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We study a hypergraph coloring invariant which was first introduced by Berge in the early 1970's and later independently in different contexts by further authors. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is a set system $\mathcal{E}$ on the underlying vertex set X. An assignment $\varphi : X \to \mathbb{N}$ is a C-coloring of $\mathcal{H}$ if each edge $E \in \mathcal{E}$ has two vertices assigned to the same number (i.e. color). Equivalently, a C-coloring is a partition of the underlying set X where no edge $E \in \mathcal{E}$ is completely sliced by the partition. The upper chromatic number $\overline{\chi}(\mathcal{H})$ of $\mathcal{H}$ is the possible maximum number of partition classes which can be achieved under this condition. We use the notation $n = |X|$ and $m = |\mathcal{E}|$ for the number of vertices and edges, respectively, in a generic input hypergraph $\mathcal{H} = (X, \mathcal{E})$.

- For the general case we prove a guaranteed approximation ratio for the difference $n - \overline{\chi}(\mathcal{H})$.

A hypertree is a hypergraph $\mathcal{H} = (X, \mathcal{E})$ for which a 'host tree' graph $T = (X, F)$ exists with the property that each edge of $\mathcal{H}$ induces a connected subgraph in T. We prove the following results on hypertrees:

- $\overline{\chi}(\mathcal{H})$ does not have an $\mathcal{O}(n^{1-\epsilon})$-approximation in polynomial time (unless $\mathsf{P} = \mathsf{NP}$).
- $\overline{\chi}(\mathcal{H})$ cannot be approximated within additive error $o(n)$ in polynomial time, even if each edge of $\mathcal{H}$ contains at most 7 vertices (unless $\mathsf{P} = \mathsf{NP}$).

Our positive result is an algorithm proving the following claim:

- The problems of determining $\overline{\chi}(\mathcal{H})$ and finding a $\overline{\chi}(\mathcal{H})$-coloring are fixed-parameter tractable in terms of maximum degree on the class of hypertrees.