Hamiltonian cycles: existence and uniqueness

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The existence of hamiltonian cycles is an NP-complete problem even in the class of planar 3-connected cubic graphs. However, there are many results on the existence of hamiltonian cycles in graphs. All of them were developed since the 1940s.

In fact, once can group some of these results by certain properties such as complete n-closures (starting with Dirac's and Ore's result and the like), or by Tutte's Bridge Theorem to prove that 4-connected planar graphs are hamiltonian. Other hamiltonian problems were solved by ad hoc methods/constructions such as dealing with EPS-graphs to prove hamiltonian cycles in the square of graphs (not just for the square of 2-connected graphs), or proving the existence of hamiltonian cycles in prisms over cubic graphs.

Not surprisingly, there are many open problems in hamiltonian graph theory. We discuss several of them such as Barnettes' Conjectures, Matthews-Sumner Conjecture as the most general conjecture in a sequence of equivalent conjectures (e.g., the Dominating Cycles Conjecture). However, we also discuss A-trails in planar eulerian graphs: their existence in eulerian triangulations of the plane is equivalent to Barnette's Conjecture on hamiltonian cycles in planar, cubic, 3-connected bipartite graphs.

In the last part of the talk, we deal with uniquely hamiltonian graphs, i.e., graphs having precisely one hamiltonian cycle. Given the fact that 3-regular graphs are not uniquely hamiltonian (C.A.B. Smith's Theorem) and the same conclusion holding for graphs without vertices of even degree (A. Thomason's Theorem), there are several open problems, the most prominent being John Sheehan's conjecture whereby 4-regular simple graphs are not uniquely hamiltonian. Note that there are uniquely hamiltonian 4-regular loopless graphs, but they have multiple edges. Recent results giving some evidence that Sheehan's conjecture might be true, deal with the automorphism group of the graphs considered. On the other hand, there exist uniquely hamiltonian graphs having 4- and 14-valent vertices only. However, the problem re. uniquely hamiltonian planar graphs is still open.