On the set of simple hypergraph degree sequences

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The problem of characterizing the set of degree sequences of simple hypergraphs is one of the known open problems in the hypergraph theory [1]-[3]. For a given m, $0 \leq m \leq 2^n$, let $\psi_m(n)$ denote the set of all degree sequences of simple hypergraphs on n vertices that have m edges. $\psi_m(n)$ can be considered as a subset of $\Xi_{m+1}^n = \{(a_1, \cdots, a_n) : 0 \leq a_i \leq m \text{ for all } i\ \}$, a ranked poset with a component-wise partial order and with the rank of an element $(a_1, \cdots, a_n)$ defined as $a_1 + \cdots + a_n$. The subject of our investigation is $\psi_m(n)$, as well as its complement in Ξ_{m+1}^n, the set of integer n-tuples that do not correspond to any degree sequence. A set-theoretical description of $\psi_m(n)$ is given in [4] but the effective algorithmic characterization of $\psi_m(n)$ is not known. Therefore, the study of the complementary area $\Xi_{m+1}^n \setminus \psi_m(n)$ can be supportive in solving the problem algorithmically.

We define $\hat{H}$, an "upper" subposet of Ξ_{m+1}^n as $\hat{H} = \{(a_1, \cdots, a_n) : m_{mid} \leq a_i \leq m$ for all i }, where $m_{mid} = m/2$ for even m, and $m_{mid} = (m+1)/2$ for odd m. Hypergraphs with degree sequences belonging to $\hat{H} \cap \psi_m(n)$, we call "upper" hypergraphs. It is known [4] that all elements of $\psi_m(n)$ can be effectively derived from the set $\hat{H} \cap \psi_m(n)$. The same is true for the complement, all elements of $\Xi_{m+1}^n \setminus \psi_m(n)$ can be found, having $\hat{H} \setminus \psi_m(n)$. Thus the problem of characterizing the set of degree sequences of simple hypergraphs is reduced from Ξ_{m+1}^n to $\hat{H}$. Furthermore, the set of degree sequences of simple "upper" hypergraphs composes an ideal $\mathcal{I}$ in $\hat{H}$, and its complement is a filter $\mathcal{F}$ in $\hat{H}$. In the process of determining all maximal elements of $\mathcal{I}$, we found the lowest r_{min} rank, as well as the highest r_{max} rank of maximal elements, finding in this manner the possible range of ranks of maximal elements in $\hat{H}$ (in Ξ_{m+1}^n). Similarly, we found the highest $\overline{r}_{max}$ rank, as well as the lowest $\overline{r}_{min}$ rank of minimal elements of $\mathcal{F}$, and determine their possible range. We obtain: all $d \in \hat{H}$ with $\sum d_i < \overline{r}_{min}$ are degree sequences of simple "upper" hypergraphs, and all $d \in \hat{H}$ with $\sum d_i > r_{max}$ are elements of $\hat{H} \setminus \psi_m(n)$. Further investigations concern the area estimates of valid ranks, depending on the value of m.

References

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