# Dichotomy of the $H$-quasi-cover problem

**Marek Tesař**

(joint work with Jiří Fiala)

A *homomorphism* between two graphs $G$ and $H$ is an edge-preserving mapping $f : V(G) \to V(H)$. We focus on homomorphisms $f$ that satisfy local constraints. For instance it might be required for each vertex $u$ of $G$ that all neighbors of its image $f(u)$, are used when the mapping $f$ is restricted on the neighborhood of $u$, formally $|f^{-1}(v) \cap N_G(u)| \geq 1$ for each $v \in N_H(f(u))$. In other words $f$ should act surjectively between $N_G(u)$ and $N_H(f(u))$ for each $u \in V(G)$. In such a situation we say that $f$ is a *locally surjective* homomorphism.

We focus in a particular case of locally surjective homomorphisms, called *quasi-coverings*. These satisfy that for every vertex $u$ of $G$ there exists a positive integer $c$ such that $|f^{-1}(v) \cap N_G(u)| = c$ for every $v \in N_H(f(u))$ — in such a case we say that $f|_{N_G(u)}$ is $c$-fold between $N_G(u)$ and $N_H(f(u))$. Note that the constant $c$ may vary for different vertices of $G$. If such a quasi-covering projection from $G$ to $H$ exists, we say that $G$ quasi-covers $H$ or that $G$ is a quasi-cover of $H$. We can define the following decision problem:

**Problem:** $H$-Quasi-cover
**Parameter:** Fixed connected graph $H$
**Input:** Connected graph $G$
**Question:** Does there exist a quasi-covering projection from $G$ to $H$?

We show that this problem is solvable in polynomial time if $H$ has at most two vertices and that it is NP-complete otherwise. As a byproduct we show constructions of regular quasi-covers and of multi-quasi-covers that might be of independent interest.